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2002 J. Phys. A: Math. Gen. 35 6883

(http://iopscience.iop.org/0305-4470/35/32/308)

Institute of Physics Publishing | Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **35** (2002) 6883–6892 | PII: S0305-4470(02)36354-6

Invariant description of solutions of hydrodynamic-type systems in hodograph space: hydrodynamic surfaces

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Received 29 April 2002, in final form 21 June 2002
Published 2 August 2002
Online at stacks.iop.org/JPhysA/35/6883

Abstract
Hydrodynamic surfaces are solutions of hydrodynamic-type systems viewed as non-parametrized submanifolds of the hodograph space. We propose an invariant differential-geometric characterization of hydrodynamic surfaces by expressing the curvature form of the characteristic web in terms of the reciprocal invariants.

PACS numbers: 02.40.Hw, 02.30.Jr
Mathematics Subject Classification: 35L40

1. Introduction

Equations of hydrodynamic type

$$u_t^i = w_j^i(u)u_x^j \qquad i, j = 1, \ldots, n \tag{1}$$

naturally arise in applications in gas dynamics, hydrodynamics, chemical kinetics, the Whitham averaging procedure, differential geometry and topological field theory. We refer to [4, 13] for their geometric theory. Any solution $u^i(x, t)$ of system (1) defines a surface in the hodograph space $u^1, \ldots, u^n$ parametrized by the independent variables x and t. Representing this surface explicitly in the form $u^3 = u^3(u^1, u^2), \ldots, u^n = u^n(u^1, u^2)$, one can readily rewrite (1) as a system of PDEs for $u^3, \ldots, u^n$ viewed as functions of u^1, u^2. For $n = 3$ this will be a third-order quasilinear PDE for $u^3(u^1, u^2)$, known as the equation of a *hydrodynamic surface* (by the term hydrodynamic surfaces we mean solutions of (1) viewed as non-parametrized submanifolds of the hodograph space). This equation was derived by Yanenko [14] and subsequently discussed in [12] in the case of one-dimensional polytropic gas dynamics (see example 4 below). For 3-component systems in Riemann invariants, the derivation of this equation is given in the appendix. Although, technically, this derivation does not cause any problems, the result is usually a rather complicated PDE. This is not surprising since the choice of, say, u^3 as a function of u^1, u^2 introduces an asymmetry in our approach from the very beginning. Thus, it is desirable to have an invariant coordinate-free description of hydrodynamic surfaces. In order to provide such a description (in what follows we concentrate on the case $n = 3$), we first have to supply the hodograph space with certain differential-geometric objects. These objects, known as the *reciprocal invariants*, are introduced in section 3. Reciprocal invariants induce further geometric structure on any hydrodynamic surface (see section 4), the characteristic 3-web being part of it. Calculating the curvature of the characteristic 3-web in terms of the reciprocal invariants, we obtain a simple invariant description of hydrodynamic surfaces (theorem 1). We apply our results to linearly degenerate semi-Hamiltonian systems in Riemann invariants (section 5) and the equations of associativity of two-dimensional topological field theory (section 6). In both cases, the characteristic 3-web proves to be *hexagonal*, thereby providing a simple coordinate-free description of hydrodynamic surfaces. We illustrate the concept of a hydrodynamic surface by concluding this introduction with a list of examples.

Example 1. The general solution of the linear system

$$R^1_t = 0 \qquad R^2_x = 0 \qquad R^3_t = R^3_x \tag{2}$$

is given by $R^1 = g^1(x)$, $R^2 = g^2(t)$, $R^3 = g^3(x+t)$ where g^i are arbitrary functions of their arguments. These formulae can be rewritten as $x = f^1(R^1), t = f^2(R^2), -x-t = f^3(R^3)$ where the functions f^i are determined by g^i. Adding these equations, we arrive at the hydrodynamic surfaces

$$f^1(R^1) + f^2(R^2) + f^3(R^3) = 0 \tag{3}$$

in the hodograph space R^1, R^2, R^3. Note that surfaces (3) solve the third-order PDE

$$\left(\ln \frac{R^3_1}{R^3_2}\right)_{12} = 0 \tag{4}$$

which is thus the equation of a hydrodynamic surface. Here subscripts denote differentiation with respect to R^1 and R^2.

Example 2. Let us consider the system

$$R^1_t = (R^2+R^3)R^1_x \qquad R^2_t = (R^1+R^3)R^2_x \qquad R^3_t = (R^1+R^2)R^3_x \tag{5}$$

which, upon the introduction of the new independent variables X and T,

$$\mathrm{d}X = \frac{\mathrm{d}x + (R^2+R^3)\,\mathrm{d}t}{(R^1-R^2)(R^1-R^3)} \qquad \mathrm{d}T = \frac{\mathrm{d}x + (R^1+R^3)\,\mathrm{d}t}{(R^2-R^1)(R^2-R^3)} \tag{6}$$

(note that these 1-forms are closed by virtue of (5)), takes the linear form (2). Transformations of this type are called *reciprocal*. Since any reciprocal transformation is just a reparametrization of solutions, hydrodynamic surfaces of both equations (2) and (5) are the same, defined by PDE (4). This example is generalized in section 5 to arbitrary 3-component linearly degenerate semi-Hamiltonian systems in Riemann invariants, which are all reciprocally related to (2) and, thus, have the same hydrodynamic surfaces.

Example 2 shows that the equation governing hydrodynamic surfaces must be expressible in terms of *reciprocal invariants*, differential-geometric objects in the hodograph space which do not change under reciprocal transformations.

Example 3. The equation of associativity of two-dimensional topological field theory

$$f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt} \tag{7}$$

Table 1. Hydrodynamic surfaces of the associativity equation.

Solution of (7)	Hydrodynamic surface in the hodograph space a, b, c
$f = \frac{x^2t^2}{4} + \frac{t^5}{60}$	Plane $a = 0$
$f = x\mathrm{e}^t - \frac{x^4}{24}$	Plane $b = 0$
$f = \frac{x^2\mathrm{e}^t}{4} + \frac{\mathrm{e}^{2t}}{32} - \frac{x^4}{48}$	Quadric $2ab + c = 0$
$f = \frac{t^2 \ln x}{2} + \frac{3t^2}{4}$	Quadric $b^2 - ac = 0$
$f = \frac{x^3t}{6} + \frac{x^2t^3}{6} + \frac{t^7}{210}$	Cayley cubic $c - 2ab + 2a^3 = 0$
$f = \frac{x^3t^2}{6} + \frac{x^2t^5}{20} + \frac{t^{11}}{3960}$	Quartic $4ac - 4a^2(b - a^2/2) = (b - a^2/2)^2$

rewritten in the variables $a = f_{xxx}, b = f_{xxt}, c = f_{xtt}$, takes the form of a 3-component system of hydrodynamic type,

$$a_t = b_x \qquad b_t = c_x \qquad c_t = (b^2 - ac)_x. \tag{8}$$

In table 1 we list the hydrodynamic surfaces (in the hodograph space a, b, c) corresponding to some explicit solutions of equation (7) as found in [3].

It would be interesting to classify the solutions of (7) whose hydrodynamic surfaces are algebraic. A differential-geometric characterization of hydrodynamic surfaces of system (8) is given in section 6.

Example 4. In Lagrangian coordinates, the equations of one-dimensional polytropic gas dynamics take the form

$$u_t + p_q = 0 \qquad \psi p^{-\kappa} p_t + u_q = 0 \qquad \psi_t = 0 \quad \kappa = \frac{\gamma + 1}{\gamma}.$$

The equation governing hydrodynamic surfaces $\psi = \psi(p, u)$ was derived in [12]. After being integrated once, it reduces to the second-order quasilinear PDE

$$(\psi p^{-\kappa} \psi_u)_u - \psi_{pp} = f(\psi) \left(\psi_p^2 - \psi p^{-\kappa} \psi_u^2\right)$$

where f is an arbitrary function of ψ.

2. Exterior representation of hydrodynamic-type systems

Let v^i be the eigenvalues of the matrix w^i_j, called the *characteristic velocities* of system (1), which we assume to be real and pairwise distinct. Let $l^i = \left(l^i_1(u), \ldots, l^i_n(u)\right)$ be the corresponding left eigenvectors, $l^i_j w^j_k = v^i l^i_k$. With the eigenforms $\omega^i = l^i_j\, \mathrm{d}u^j$, the system (1) is readily rewritten in the exterior form,

$$\omega^i \wedge (\mathrm{d}x + v^i \mathrm{d}t) = 0 \qquad i = 1, \ldots, n. \tag{9}$$

Differentiation of ω^i and v^i gives the *structure equations*

$$\mathrm{d}\omega^i = -c^i_{jk} \omega^j \wedge \omega^k \qquad \left(c^i_{jk} = -c^i_{kj}\right) \qquad \mathrm{d}v^i = v^i_j \omega^j \tag{10}$$

containing all the necessary information about the system under study. For diagonalizable systems we have $c^i_{jk} = 0$, so that $\omega^i = \mathrm{d}R^i$, where the variables R^i are called the *Riemann invariants* of system (1). In Riemann invariants, equations (9) take a diagonal form,

$$R^i_t = v^i(R) R^i_x \qquad i = 1, \ldots, n. \tag{11}$$

In this case $v^i_j = \partial v^i / \partial R^j$.

3. Reciprocal transformations and reciprocal invariants

We recall the necessary information about reciprocal transformations of hydrodynamic-type systems. Let $B(u)\,\mathrm{d}x + A(u)\,\mathrm{d}t$ and $N(u)\,\mathrm{d}x + M(u)\,\mathrm{d}t$ be two conservation laws of system (1), understood as the 1-forms which are closed by virtue of (1). In the new independent variables X, T defined by

$$\mathrm{d}X = B(u)\,\mathrm{d}x + A(u)\,\mathrm{d}t \qquad \mathrm{d}T = N(u)\,\mathrm{d}x + M(u)\,\mathrm{d}t \tag{12}$$

the system (1) takes the form

$$u^i_T = W^i_j(u)u^j_X \tag{13}$$

where $W = (Bw - AE)(ME - Nw)^{-1}$, $E = \mathrm{id}$. The new characteristic velocities V^i are

$$V^i = \frac{v^i B - A}{M - v^i N} \tag{14}$$

while the eigenforms ω^i remain the same. Transformations (12) are called *reciprocal*. We refer to [11] for the discussion of their applications in gas dynamics, hydrodynamics and soliton theory. In [6–8] the author introduced a number of objects in the hodograph space which prove to be reciprocally invariant. In the 3-component case these are

(1) Three two-dimensional characteristic distributions:

$$\omega^i = 0 \qquad i = 1, 2, 3. \tag{15}$$

Note that the eigenforms ω^i are only defined up to a nonzero multiple ($\omega^i \to p^i\omega^i$), so that only these distributions make an invariant sense. It is natural to call these distributions *characteristic* since, by virtue of (9), characteristic directions are the intersections of a tangent plane to a hydrodynamic surface with the distributions (15).

(2) The 1-forms:

$$\Omega^1 = \frac{v^1_1(v^2 - v^3)}{(v^1 - v^2)(v^1 - v^3)}\omega^1 \qquad \Omega^2 = \frac{v^2_2(v^3 - v^1)}{(v^2 - v^1)(v^2 - v^3)}\omega^2$$
$$\Omega^3 = \frac{v^3_3(v^1 - v^2)}{(v^3 - v^1)(v^3 - v^2)}\omega^3. \tag{16}$$

In the case $v^i_i \neq 0$ these 1-forms contain the information about the distributions (15). We really need (15) only when some of the v^i_i are zero.

(3) The differential $\mathrm{d}\Omega$ of the 1-form:

$$\Omega = \left(\frac{v^2_1 - \frac{1}{2}v^1_1}{v^1 - v^2} + \frac{v^3_1 - \frac{1}{2}v^1_1}{v^1 - v^3}\right)\omega^1 + \left(\frac{v^1_2 - \frac{1}{2}v^2_2}{v^2 - v^1} + \frac{v^3_2 - \frac{1}{2}v^2_2}{v^2 - v^3}\right)\omega^2$$
$$+ \left(\frac{v^1_3 - \frac{1}{2}v^3_3}{v^3 - v^1} + \frac{v^2_3 - \frac{1}{2}v^3_3}{v^3 - v^2}\right)\omega^3 \tag{17}$$

(note that Ω itself is not reciprocally invariant).

Remark. To prove the reciprocal invariance of these objects, it is sufficient to consider reciprocal transformations of the two simpler types, namely

(a) The interchange of the independent variables x and t ($A = N = 1$, $B = M = 0$ in (12), implying $V^i = 1/v^i$).

(b) Transformations which preserve the variable t ($N = 0$, $M = 1$ implying $V^i = v^i B - A$). In both cases, the invariance of (16) and (17) is a result of a simple calculation. In the case (b) one has to use the identities $A_i = v^i B_i$ (no summation!) which characterize conservation laws of hydrodynamic-type systems. Here A_i and B_i are defined by the expansions $\mathrm{d}A = A_i \omega^i$, $\mathrm{d}B = B_i \omega^i$. Since an arbitrary reciprocal transformation is a superposition of transformations (a) and (b), the statement follows.

Although the set of reciprocal invariants (16) and (17) is complete [6, 8], further invariants can be easily constructed. For instance, the pseudo-Riemannian metric

$$\Omega^1\Omega^2 + \Omega^1\Omega^3 + \Omega^2\Omega^3 = \frac{v_1^1 v_2^2}{(v^1 - v^2)^2}\omega^1\omega^2 + \frac{v_1^1 v_3^3}{(v^1 - v^3)^2}\omega^1\omega^3 + \frac{v_2^2 v_3^3}{(v^2 - v^3)^2}\omega^2\omega^3$$

is also reciprocally invariant. The Lie-geometric interpretation of reciprocal invariants was proposed in [10]. Note that for *linearly degenerate* systems (that is, for systems with $v_i^i = 0$ for any i), some of the reciprocal invariants vanish. The only ones that survive are the 2-form $\mathrm{d}\Omega$ and the characteristic distributions (which are not involutive unless the system possesses Riemann invariants). Reciprocal transformations are known to preserve the linear degeneracy.

4. Geometry of hydrodynamic surfaces

The following objects are naturally induced on hydrodynamic surfaces of 3-component systems of hydrodynamic type.

The characteristic 3-web is a collection of three 1-parameter families of curves defined by the equations $\omega^i = 0$. Geometrically, characteristic directions are intersections of the tangent plane of a hydrodynamic surface with the two-dimensional characteristic distributions $\omega^i = 0$ in the hodograph space. For a 3-web on a surface, let ϕ^1 and ϕ^2 be the 1-forms such that the curves of the first, second and third families are defined by the equations $\phi^1 = 0$, $\phi^2 = 0$ and $\phi^1 = \phi^2$, respectively (actually, ϕ^1 and ϕ^2 are the properly normalized ω^1 and ω^2; their choice is obviously non-unique since they can be multiplied by a common factor). The connection form ϕ of a 3-web is uniquely determined by the exterior equations

$$\mathrm{d}\phi^1 = \phi \wedge \phi^1 \qquad \mathrm{d}\phi^2 = \phi \wedge \phi^2.$$

Finally, the curvature 2-form C equals $\mathrm{d}\phi$. It has an invariant meaning and does not depend on the particular normalization of ϕ^1 and ϕ^2. For instance, in the case of 3-component systems (11) in Riemann invariants we have $\omega^i = \mathrm{d}R^i$ so that the characteristics are defined by $\mathrm{d}R^1 = 0$, $\mathrm{d}R^2 = 0$ and $\mathrm{d}R^3 = R_1^3\,\mathrm{d}R^1 + R_2^3\,\mathrm{d}R^2 = 0$ (the hydrodynamic surface is parametrized in the form $R^3(R^1, R^2)$). Clearly, one can choose $\phi^1 = R_1^3\,\mathrm{d}R^1$ and $\phi^2 = -R_2^3\,\mathrm{d}R^2$, implying that the connection form is

$$\phi = \left(\ln R_1^3\right)_2 \mathrm{d}R^2 + \left(\ln R_2^3\right)_1 \mathrm{d}R^1.$$

The corresponding curvature 2-form is

$$C = \mathrm{d}\phi = \left(\ln R_1^3 - \ln R_2^3\right)_{12} \mathrm{d}R^1 \wedge \mathrm{d}R^2.$$

We will use this expression when deriving the equation of hydrodynamic surfaces in the appendix. Recall that the zero curvature webs are called hexagonal [2].

The form $\mathrm{d}\Omega$ is just the restriction of the reciprocal invariant $\mathrm{d}\Omega$ to a hydrodynamic surface.

The forms $*\Omega^i$ are defined as follows. Take, say, the reciprocally invariant form Ω^1 and restrict it to a hydrodynamic surface. On the same hydrodynamic surface, choose the metric

$2\,\mathrm{d}R^2\,\mathrm{d}R^3$ with the volume form $\mathrm{d}R^2 \wedge \mathrm{d}R^3$ (note the important order of indices 2 and 3 in the expressions for Ω^1 and the volume form). Construct the vector which is dual to the 1-form Ω^1 with respect to the metric chosen. Finally, evaluate the volume form on this vector. The result will be a 1-form which is usually denoted by $*\Omega^1$. Note that in two dimensions the $*$-operator is conformally invariant, so that only the conformal class of the metric $2\,\mathrm{d}R^2\,\mathrm{d}R^3$ matters. The ambiguity in the choice of the order of indices 2 and 3 in the expression for Ω^1, combined with the ambiguity in choosing the sign (orientation) of the volume form, gives a well-defined answer for $*\Omega^1$. In general, for the 1-form

$$\Omega^i = \frac{v^i_i(v^j - v^k)}{(v^i - v^j)(v^i - v^k)}\,\mathrm{d}R^i$$

the 1-form $*\Omega^i$ is defined by choosing the metric $2\,\mathrm{d}R^j\,\mathrm{d}R^k$ with the volume form $\mathrm{d}R^j \wedge \mathrm{d}R^k$. The $*$-operator is invariant and can be applied in any convenient coordinate system leading to one and the same result. The relevant computation is shown below in the proof of theorem 1.

Now we are in the position to formulate the main result of this paper.

Theorem 1. *The curvature 2-form C of the characteristic 3-web is given by the formula*

$$C = -\mathrm{d}\Omega - \tfrac{1}{2}\,\mathrm{d}(*\Omega^1 + *\Omega^2 + *\Omega^3). \tag{18}$$

Written down in any suitable coordinate system in the hodograph space, equation (18) reduces to a third-order PDE for hydrodynamic surfaces.

Proof. Note that equation (18) is manifestly coordinate-free. Thus, it suffices to establish (18) in any local parametrization of a hydrodynamic surface. We will work directly in the coordinates x, t. For simplicity, we assume that the system in question possesses Riemann invariants. This assumption is not important and the general proof is literally the same.

Introducing the 1-forms

$$\phi^1 = (v^2 - v^3)(\mathrm{d}x + v^1\,\mathrm{d}t) \qquad \text{and} \qquad \phi^2 = (v^1 - v^3)(\mathrm{d}x + v^2\,\mathrm{d}t)$$

we readily see that the characteristics of the first, second and third families are defined by the equations $\phi^1 = 0$, $\phi^2 = 0$ and $\phi^1 = \phi^2$, respectively. The connection form ϕ of the 3-web is uniquely determined by the exterior equations

$$\mathrm{d}\phi^1 = \phi \wedge \phi^1 \qquad \mathrm{d}\phi^2 = \phi \wedge \phi^2. \tag{19}$$

With $\phi = a\,\mathrm{d}x + b\,\mathrm{d}t$, equation $(19)_1$ gives

$$b - av^1 = -v^1_1 R^1_x + \left((v^2 - v^1)\frac{v^2_2 - v^3_2}{v^2 - v^3} - v^1_2\right) R^2_x + \left((v^3 - v^1)\frac{v^2_3 - v^3_3}{v^2 - v^3} - v^1_3\right) R^3_x.$$

Similarly, equation $(19)_2$ implies

$$b - av^2 = -v^2_2 R^2_x + \left((v^1 - v^2)\frac{v^1_1 - v^3_1}{v^1 - v^3} - v^2_1\right) R^1_x + \left((v^3 - v^2)\frac{v^1_3 - v^3_3}{v^1 - v^3} - v^2_3\right) R^3_x.$$

Solving for a and b, we obtain

$$a = \left(\frac{v^2_1 - v^1_1}{v^2 - v^1} + \frac{v^3_1 - v^1_1}{v^3 - v^1}\right) R^1_x + \left(\frac{v^1_2 - v^2_2}{v^1 - v^2} + \frac{v^3_2 - v^2_2}{v^3 - v^2}\right) R^2_x + \left(\frac{v^1_3 - v^3_3}{v^1 - v^3} + \frac{v^2_3 - v^3_3}{v^2 - v^3}\right) R^3_x$$

$$b = \left(\frac{v^1 v^2_1 - v^2 v^1_1}{v^2 - v^1} + v^1\frac{v^3_1 - v^1_1}{v^3 - v^1}\right) R^1_x + \left(\frac{v^2 v^1_2 - v^1 v^2_2}{v^1 - v^2} + v^2\frac{v^3_2 - v^2_2}{v^3 - v^2}\right) R^2_x$$
$$+ \left(v^3\frac{v^1_3 - v^3_3}{v^1 - v^3} + v^3\frac{v^2_3 - v^3_3}{v^2 - v^3} - v^3_3\right) R^3_x.$$

Using the identities $\mathrm{d}R^i = R^i_x(\mathrm{d}x + v^i\,\mathrm{d}t)$, it is now a direct algebraic calculation to verify that

$$a\,\mathrm{d}x + b\,\mathrm{d}t = -\Omega + \frac{1}{2}v^1_1\left(\frac{\mathrm{d}R^1}{v^1 - v^2} + \frac{\mathrm{d}R^1}{v^1 - v^3} - 2R^1_x\,\mathrm{d}t\right) + \frac{1}{2}v^2_2\left(\frac{\mathrm{d}R^2}{v^2 - v^1} + \frac{\mathrm{d}R^2}{v^2 - v^3} - 2R^2_x\,\mathrm{d}t\right) + \frac{1}{2}v^3_3\left(\frac{\mathrm{d}R^3}{v^3 - v^1} + \frac{\mathrm{d}R^3}{v^3 - v^2} - 2R^3_x\,\mathrm{d}t\right) \tag{20}$$

where the last three terms are nothing but $-\frac{1}{2}(*\Omega^1)$, $-\frac{1}{2}(*\Omega^2)$ and $-\frac{1}{2}(*\Omega^3)$, respectively. Indeed, let us calculate $*\Omega^1$. In coordinates x, t the 1-form $\Omega^1 = v^1_1(v^2 - v^3)\,\mathrm{d}R^1/(v^1 - v^2)(v^1 - v^3)$ has components

$$\left(\mu R^1_x, \mu v^1 R^1_x\right) \tag{21}$$

where $\mu = v^1_1(v^2 - v^3)/(v^1 - v^2)(v^1 - v^3)$. The metric used to define the $*$-operator is $2\,\mathrm{d}R^2\,\mathrm{d}R^3 = 2R^2_xR^3_x(\mathrm{d}x^2 + (v^2 + v^3)\,\mathrm{d}x\,\mathrm{d}t + v^2v^3\,\mathrm{d}t^2)$ or, in matrix form,

$$R^2_xR^3_x\begin{pmatrix} 2 & v^2 + v^3 \\ v^2 + v^3 & 2v^2v^3 \end{pmatrix} \tag{22}$$

with the inverse

$$\frac{1}{R^2_xR^3_x(v^3 - v^2)^2}\begin{pmatrix} -2v^2v^3 & v^2 + v^3 \\ v^2 + v^3 & -2 \end{pmatrix}. \tag{23}$$

The corresponding volume 2-form is

$$\mathrm{d}R^2 \wedge \mathrm{d}R^3 = R^2_xR^3_x(v^3 - v^2)\,\mathrm{d}x \wedge \mathrm{d}t. \tag{24}$$

Multiplying (21) by (23), we obtain the vector (the dual of Ω^1)

$$\frac{\mu R^1_x}{R^2_xR^3_x(v^3 - v^2)^2}(v^1(v^2 + v^3) - 2v^2v^3, v^2 + v^3 - 2v^1). \tag{25}$$

Finally, evaluating the volume 2-form (24) on the vector (25), we obtain the 1-form

$$*\Omega^1 = \frac{\mu R^1_x}{v^3 - v^2}((v^1(v^2 + v^3) - 2v^2v^3)\,\mathrm{d}t - (v^2 + v^3 - 2v^1)\,\mathrm{d}x)$$

which, after a simple rearrangement of terms, can be rewritten as

$$*\Omega^1 = v^1_1\left(2R^1_x\,\mathrm{d}t - \frac{\mathrm{d}R^1}{v^1 - v^2} - \frac{\mathrm{d}R^1}{v^1 - v^3}\right).$$

Comparison with (20) and the exterior differentiation complete the proof. □

In the nondiagonalizable case, the proof is essentially the same. The only difference is that we have to write $\omega^i = p^i(\mathrm{d}x + v^i\,\mathrm{d}t)$ instead of $\mathrm{d}R^i = R^i_x(\mathrm{d}x + v^i\,\mathrm{d}t)$ and to replace R^i_x by p^i in all places where they appear. Another proof (which is more constructive, although it only applies to systems in Riemann invariants) is given in the appendix.

5. Linearly degenerate semi-Hamiltonian systems in Riemann invariants

A system of hydrodynamic type in Riemann invariants,

$$R^i_t = v^i(R)R^i_x$$

is called *linearly degenerate* if $v^i_i = 0$ for any i. It is called *semi-Hamiltonian* if

$$\left(\frac{v^i_j}{v^j - v^i}\right)_k = \left(\frac{v^i_k}{v^k - v^i}\right)_j$$

for any $i \neq j \neq k$. The last condition is equivalent to the existence of an infinity of conservation laws and hydrodynamic symmetries and the integrability of the system under study by the *generalized hodograph transform* [13]. One can readily verify that the reciprocal invariants $d\Omega$ and Ω^1, Ω^2, Ω^3 of 3-component linearly degenerate semi-Hamiltonian systems (the example of which is (5)) are zero. All such systems can be linearized by a reciprocal transformation, and hydrodynamic surfaces thereof are governed by one and the same PDE (4). Geometrically, hydrodynamic surfaces are uniquely characterized as the surfaces in the hodograph space R^1, R^2, R^3 on which the 3-web, cut by the coordinate planes $R^i = \text{const}$, is hexagonal. We refer to [5, 7] for a further discussion of the geometry of characteristic webs on solutions of linearly degenerate semi-Hamiltonian systems.

6. Equations of associativity

It was demonstrated in [9] that system (8) can be transformed by a reciprocal transformation into a system with constant characteristic velocities. Since for systems with constant characteristic velocities the objects Ω^1, Ω^2, Ω^3 and Ω are automatically zero, they are zero for system (8) as well (in view of their reciprocal invariance). Thus, theorem 1 implies that hydrodynamic surfaces of (8) are uniquely characterized as surfaces on which the characteristic 3-web (cut by the characteristic distributions (15)) is hexagonal. These distributions have a simple algebro-geometric description which we briefly discuss below (see also [1]).

In the hodograph space a, b, c consider the twisted cubic γ,

$$a = -3t \qquad b = -3t^2/2 \qquad c = -t^3.$$

For any point p in the hodograph space, there are exactly three osculating planes of γ containing p. In each of these planes, draw a line through p parallel to the tangential direction to γ in the point where this plane osculates γ. Thus, one obtains three lines through each point p in the hodograph space. These lines are the *rarefaction curves* of system (8). The three two-dimensional characteristic distributions in question are spanned by each pair thereof.

This construction can be reformulated in a projectively invariant way as follows. Consider a twisted cubic γ in the projective space P^3 and fix the plane Λ which osculates γ (in the construction above this was the plane at infinity). Take any other plane π which osculates γ in a point p. The tangent line to γ in the point p cuts Λ in the point $A(p)$ (as p varies, the collection of points $A(p)$ is a conic in Λ). Finally, consider a pencil of lines in the plane π with the vertex in $A(p)$. As p varies, this gives a 2-parameter family, or a congruence of lines in P^3, which is of the order 3 (that is, there are precisely 3 lines of the congruence through a generic point of P^3). The three two-dimensional distributions are spanned by each pair thereof. In the case when Λ is the plane at infinity, this construction reduces to that described above. Hydrodynamic surfaces in question are those on which these two-dimensional distributions cut a hexagonal 3-web. These considerations and example 3 clearly indicate that it is of interest to classify hydrodynamic surfaces which are algebraic.

Note that this problem makes sense for arbitrary congruences of the order 3 in P^3, since any such congruence induces a 3-web on a surface.

Acknowledgment

This research was supported by the EPSRC grant no Gr/N30941.

Appendix. Another proof of theorem 1

We will derive the equation of a hydrodynamic surface for 3-component systems in Riemann invariants assuming $R^3 = R^3(R^1, R^2)$. We use the notation R^3_1, R^3_2 for partial derivatives of R^3 with respect to R^1 and R^2 which are now viewed as independent variables. By virtue of (11), one has

$$\mathrm{d}R^1 = p^1(\mathrm{d}x + v^1\mathrm{d}t) \qquad \mathrm{d}R^2 = p^2(\mathrm{d}x + v^2\mathrm{d}t) \qquad \mathrm{d}R^3 = p^3(\mathrm{d}x + v^3\mathrm{d}t) \tag{26}$$

(here $p^i = R^i_x$). On the other hand,

$$\mathrm{d}R^3 = R^3_1\,\mathrm{d}R^1 + R^3_2\,\mathrm{d}R^2 = R^3_1 p^1(\mathrm{d}x + v^1\mathrm{d}t) + R^3_2 p^2(\mathrm{d}x + v^2\,\mathrm{d}t)$$

implying

$$R^3_1 p^1 + R^3_2 p^2 = p^3 \qquad R^3_1 p^1 v^1 + R^3_2 p^2 v^2 = p^3 v^3$$

so that

$$p^1 = \frac{v^2 - v^3}{v^2 - v^1}\frac{p^3}{R^3_1} \qquad p^2 = \frac{v^1 - v^3}{v^1 - v^2}\frac{p^3}{R^3_2}.$$

Substituting these expressions into the first two equations (26), we obtain

$$\mathrm{d}x + v^1\mathrm{d}t = \frac{v^2 - v^1}{v^2 - v^3}\frac{R^3_1}{p^3}\,\mathrm{d}R^1 \qquad \mathrm{d}x + v^2\mathrm{d}t = \frac{v^1 - v^2}{v^1 - v^3}\frac{R^3_2}{p^3}\,\mathrm{d}R^2$$

so that

$$\mathrm{d}t = \frac{R^3_1}{v^3 - v^2}\frac{1}{p^3}\,\mathrm{d}R^1 + \frac{R^3_2}{v^3 - v^1}\frac{1}{p^3}\,\mathrm{d}R^2 \qquad \mathrm{d}x = \frac{R^3_1}{v^2 - v^3}\frac{v^2}{p^3}\,\mathrm{d}R^1 + \frac{R^3_2}{v^1 - v^3}\frac{v^1}{p^3}\,\mathrm{d}R^2.$$

Introducing g by the formula $1/p^3 = \mathrm{e}^g(v^3 - v^1)(v^3 - v^2)$, we ultimately have

$$\begin{aligned}\mathrm{d}t &= \mathrm{e}^g\left(R^3_1(v^3 - v^1)\,\mathrm{d}R^1 + R^3_2(v^3 - v^2)\,\mathrm{d}R^2\right)\\ \mathrm{d}x &= \mathrm{e}^g\left(R^3_1(v^1 - v^3)v^2\,\mathrm{d}R^1 + R^3_2(v^2 - v^3)v^1\,\mathrm{d}R^2\right).\end{aligned} \tag{27}$$

With $\mathrm{d}g = g_1\,\mathrm{d}R^1 + g_2\,\mathrm{d}R^2$ the differentiation of $(27)_1$ implies

$$\begin{aligned}&g_1 R^3_2(v^3 - v^2) - g_2 R^3_1(v^3 - v^1) + R^3_{12}(v^1 - v^2) + R^3_2\left(v^3_1 - v^2_1\right) + R^3_1\left(v^3_2 - v^1_2\right)\\ &\qquad + R^3_1 R^3_2\left(v^1_3 - v^2_3\right) = 0.\end{aligned}$$

Similarly, the differentiation of $(27)_2$ gives

$$\begin{aligned}&g_1 R^3_2(v^3 - v^2)v^1 - g_2 R^3_1(v^3 - v^1)v^2 + R^3_{12}(v^1 - v^2)v^3 + R^3_2\left(v^1\left(v^3_1 - v^2_1\right) + v^1_1(v^3 - v^2)\right)\\ &\qquad + R^3_1\left(v^2\left(v^3_2 - v^1_2\right) + v^2_2(v^3 - v^1)\right) + R^3_1 R^3_2\left(v^3\left(v^1_3 - v^2_3\right) + (v^1 - v^2)v^3_3\right) = 0.\end{aligned}$$

Solving these equations for g_1 and g_2, one readily obtains

$$-g_1 = \frac{R^3_{12}}{R^3_2} + \frac{v^1_1}{v^1 - v^2} + \frac{v^3_1 - v^2_1}{v^3 - v^2} + \frac{R^3_1}{R^3_2}\frac{v^2_2(v^1 - v^3)}{(v^2 - v^1)(v^2 - v^3)} + R^3_1\left(\frac{v^3_3}{v^3 - v^2} + \frac{v^1_3 - v^2_3}{v^1 - v^2}\right)$$

and

$$-g_2 = \frac{R^3_{12}}{R^3_1} + \frac{v^2_2}{v^2 - v^1} + \frac{v^3_2 - v^1_2}{v^3 - v^1} + \frac{R^3_2}{R^3_1}\frac{v^1_1(v^2 - v^3)}{(v^1 - v^2)(v^1 - v^3)} + R^3_2\left(\frac{v^3_3}{v^3 - v^1} + \frac{v^1_3 - v^2_3}{v^1 - v^2}\right)$$

or, in differential form,

$$\begin{aligned}-\mathrm{d}g &= \frac{R^3_{12}}{R^3_2}\,\mathrm{d}R^1 + \frac{R^3_{12}}{R^3_1}\,\mathrm{d}R^2 + \frac{v^1_1}{v^1 - v^2}\,\mathrm{d}R^1 + \frac{v^2_2}{v^2 - v^1}\,\mathrm{d}R^2 + \frac{v^3_1 - v^2_1}{v^3 - v^2}\,\mathrm{d}R^1 + \frac{v^3_2 - v^1_2}{v^3 - v^1}\,\mathrm{d}R^2\\ &\quad + \frac{v^1_3 - v^2_3}{v^1 - v^2}\,\mathrm{d}R^3 + \frac{R^3_1}{R^3_2}\frac{v^2_2(v^1 - v^3)}{(v^2 - v^1)(v^2 - v^3)}\,\mathrm{d}R^1 + \frac{R^3_2}{R^3_1}\frac{v^1_1(v^2 - v^3)}{(v^1 - v^2)(v^1 - v^3)}\,\mathrm{d}R^2\\ &\quad + R^3_1\frac{v^3_3}{v^3 - v^2}\,\mathrm{d}R^1 + R^3_2\frac{v^3_3}{v^3 - v^1}\,\mathrm{d}R^2.\end{aligned}$$

It is now a direct algebraic calculation to verify that terms in the last equation can be rearranged in such a way that it becomes

$$-\mathrm{d}g - \mathrm{d}\ln(v^1 - v^2)(v^1 - v^3)(v^2 - v^3) = \left(\frac{R^3_{12}}{R^3_2}\,\mathrm{d}R^1 + \frac{R^3_{12}}{R^3_1}\,\mathrm{d}R^2\right) + \Omega$$
$$+ \frac{v^1_1(v^2 - v^3)}{(v^1 - v^2)(v^1 - v^3)}\left(\frac{1}{2}\,\mathrm{d}R^1 + \frac{R^3_2}{R^3_1}\,\mathrm{d}R^2\right) + \frac{v^2_2(v^1 - v^3)}{(v^2 - v^1)(v^2 - v^3)}$$
$$\times\left(\frac{1}{2}\,\mathrm{d}R^2 + \frac{R^3_1}{R^3_2}\,\mathrm{d}R^1\right) + \frac{1}{2}\frac{v^3_3(v^1 - v^2)}{(v^3 - v^1)(v^3 - v^2)}\left(R^3_2\,\mathrm{d}R^2 - R^3_1\,\mathrm{d}R^1\right). \tag{28}$$

Now, the first term on the right,

$$\frac{R^3_{12}}{R^3_2}\,\mathrm{d}R^1 + \frac{R^3_{12}}{R^3_1}\,\mathrm{d}R^2$$

is the connection form of the characteristic 3-web; its differential is the curvature form of the web. The form Ω is defined in section 3 (recall that its differential is reciprocally invariant). The last three forms on the right are nothing but $*\Omega^1/2$, $*\Omega^2/2$ and $*\Omega^3/2$, respectively. Finally, the differentiation of (28) completes the proof.

Note that this proof is constructive: once a hydrodynamic surface is given, it can be parametrized by the independent variables t, x according to formulae (27), where g is given by (28). This parametrization is unique up to the obvious symmetries $x \to cx + a$, $t \to ct + b$ where a, b and c are constants.

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